Last Time: Injectivity + Surjectivity.

Recall: $f: A \to B$ is...

injective when $\forall x, y \in A \; [f(x) = f(y) \to x = y]$.

surjective when $\forall y \in B \; \exists x \in A \; [f(x) = y]$

bijective when it is both injective and surjective.

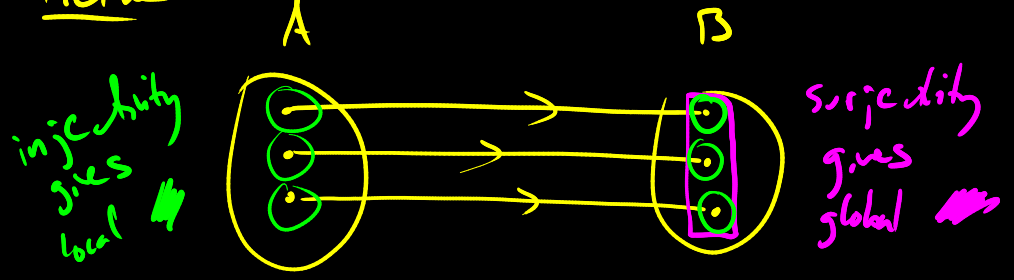
Remark: Bijections are also called "one-to-one correspondences"

If $f: A \to B$ is a bijection, then:

Injectivity $\rightsquigarrow$ distinct elts map to distinct elts...

Surjectivity $\rightsquigarrow$ every elt of B is "captured".

Picture




Prop: Two sets A and B have the same cardinality if and only if there is a bijection $A \xrightarrow{f} B$.

↳ Correspondence principle "

Ex: The function $f: \mathbb{Z} \to \mathbb{Z}$ w/ $f(n) = 3n + 7$ is Not bijective. Note f is injective:

$f(x) = f(y)$ iff $3x + 7 = 3y + 7$ iff $3x = 3y$ iff $x = y$

However, $[f(n) = 3n + 7 \equiv 1 \mod 3]$ for all $n \in \mathbb{Z}$.

In particular, $f(n) \neq 0$ for all $n \in \mathbb{Z}$, but $0 \equiv 1 \mod 3$ (absurd!). Thus $f$ is NOT surjective, so it can't be bijective... ☒

Ex: For all sets $S$, $\mathrm{id}_S$ is a bijection. ☒

Ex: The function $f: \mathbb{Z} \to \mathbb{Z}$ w/ $f(n) = n+4$ is bijective. In particular:

$f(x) = f(y)$ iff $x+4 = y+4$ iff $x = y$ for all $x, y \in \mathbb{Z}$

Given $y \in \mathbb{Z}$, $\boxed{y-4 \in \mathbb{Z}}$ and thus $f(y-4) = (y-4)+4 = y$ yields surjectivity. ↑ (crucial) Hence $f$ is bijective ☒

Ex: $f: \mathbb{N}_0 \to \mathbb{N}_0$ w/ $f(n) = n+2$ is injective.

$f$ is not surjective, because $f(n) = n+2 \geq 2$ for all $n \geq 0$. Thus $0 \notin \mathrm{ran}(f)$.

Thus $f$ is not bijective.

Ex: $f: \mathbb{Z} \to \mathbb{N}_0$ w/ $f(n) = |n|$ is surjective, but not injective. Indeed, $f(-3) = |-3| = |3| = f(3)$ but $3 \neq -3$, so $f$ is not injective. ☒

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Defn: Let $f$ be a function.

① The image of $S \subseteq \mathrm{dom}(f)$ under $f$ is

$$f(S) = \{f(s) : s \in S\}.$$

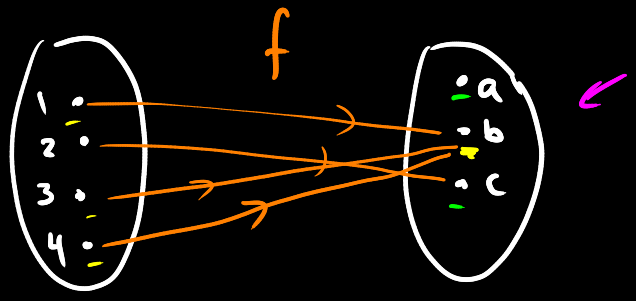
② The range of $f$ is $\mathrm{ran}(f) = f(\mathrm{dom}(f))$.

**Prop:** The function $f$ is surjective when $\text{ran}(f) = \text{cod}(f)$.

↳ Proof is an exercise!

**Defn:** Let $f$ be a function. The preimage of a set $S \subseteq \text{cod}(f)$ is $f^{-1}(S) = \{x \in \text{dom}(f) : f(x) \in S\}$.

**Ex:**




For $S = \{1,3\}$ : $f(S) = \{f(1), f(3)\} = \{b\}$

For $S = \{2,4\}$ : $f(S) = \{f(2), f(4)\} = \{c, b\}$

(Observation: $\#f(S) \leq \#S$). ←

For $T = \{a,c\}$: $f^{-1}(T) = \{2\}$

$T = \{a\}$ : $f^{-1}(T) = \emptyset$

$T = \{b\}$ : $f^{-1}(T) = \{1,3,4\}$

NB: NO correlation between $\#T$ and $\#f^{-1}(T)$!

(Observation: $f^{-1}(\text{cod}(f)) = \text{dom}(f)$).

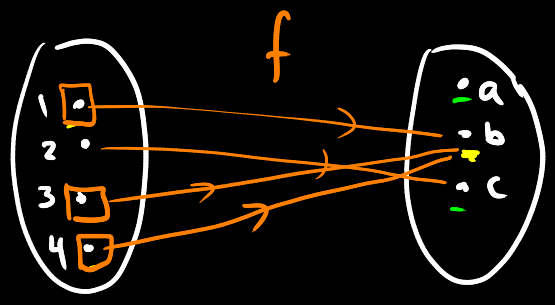
Really saw: $f^{-1}(\text{ran}(f)) = \text{dom}(f)$

$\text{ran}(f) = f(\text{dom}(f))$

Q: What is $f^{-1}(f(S))$ for $S \subseteq \text{dom}(f)$?

A: Don't know exactly...

Ex:




$S = \{1,3\}$

$f(S) = \{b\}$

$f^{-1}(f(S)) = f^{-1}\{b\} = \{1,3,4\}$

**Prop:** For all functions $f$ and $S \subseteq \text{dom}(f)$,

$$f^{-1}(f(S)) \supseteq S.$$

**Pf:** Let $f$ be a function and $S \subseteq \text{dom}(f)$. Let $x \in S$ be arbitrary. By definition of $f(S)$ we have $f(x) \in f(S)$. On the other hand, $f^{-1}(f(S)) = \{z \in \text{dom}(f) : f(z) \in f(S)\}$. As $x \in S$ and $S \subseteq \text{dom}(f)$ we have $x \in \text{dom}(f)$. Moreover, $f(x) \in f(S)$ yields $x \in f^{-1}(f(S))$ by definition of preimage. As $x \in S$ was arbitrary, we have $S \subseteq f^{-1}(f(S))$ as desired. ☒

**Prop:** If $f$ is an injective function, then $f^{-1}(f(S)) = S$ for every $S \subseteq \text{dom}(f)$. Conversely, if $f^{-1}(f(S)) = S$ for all $S \subseteq \text{dom}(f)$, then $f$ is injective.

**pf:** Let $f$ be a function.

($\Rightarrow$): Let $S \subseteq \text{dom}(f)$ be arbitrary and suppose $f$ is injective. By the previous proposition, $S \subseteq f^{-1}(f(S))$. Let

$x \in f^{-1}(f(S))$ be arbitrary. (WTS: $x \in S$).

By definition of the preimage, we have $f(x) \in f(S)$.

So $[f(x) = f(y)]$ for some $y \in S$ by definition of the image. Hence $x = y$ because $f$ is injective. Thus $x \in S$. As $x \in f^{-1}(f(S))$ was arbitrary, we've shown $f^{-1}(f(S)) \subseteq S$.

($\Leftarrow$): Assume $f^{-1}(f(S)) = S$ for all $S \subseteq \text{dom}(f)$.

(WTS: $f$ is injective). Let $x, y \in \text{dom}(f)$ be arbitrary and assume $f(x) = f(y)$. In particular, $\{x\} = f^{-1}(f\{x\}) = f^{-1}\{f(x) = f(y)\} = f^{-1}(f\{y\}) = \{y\}$ by our assumptions. Hence $x \in \{y\}$ yields $x = y$.

Hence $f$ is injective, by definition. ∎

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There are analogous propositions involving surjectivity...

Prop: Let $f$ be a function. The function $f$ is surjective if and only if for all $T \subseteq \text{cod}(f)$ we have $f(f^{-1}(T)) = T$.

Lem: For all functions $f$ and all $T \subseteq \text{cod}(f)$,

$$f(f^{-1}(T)) \subseteq T.$$

Exercise: Prove it!